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EXTENDING MONOTONE DECOMPOSITIONS OF 3-MANIFOLDS

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1. **Introduction.** Suppose X is a closed subset of Euclidean 3-space E^3 and f is a compact monotone map of X. We shall be concerned with conditions under which f may be extended to a compact monotone map of E^3 such that the image space is topologically E^3.

It was our original intention to show that if f has only a countable number of point inverses and none of these separates E^3, then there is a monotone upper semicontinuous decomposition of E^3 onto itself with the point inverses being elements of the decomposition. As we developed tools for proving this, we found that these tools proved more. Since this paper was written we learn that Ralph J. Bean studies similar situations in a paper entitled "Repairing embeddings and decompositions in S^3" to appear in Duke Mathematical Journal.

We find it convenient to state our preliminary results for compact 3-manifolds, so in first sections of this paper we deal with the 3-sphere S^3 rather than E^3. We suppose that E^3 and S^3 have their customary rectilinear structures.

By an *n-manifold* we mean a separable metric space which is locally like Euclidean n-space. We do not imply without specification that it is either connected, compact, triangulated, or orientable. However, we do imply that it is without boundary. If we want to permit the possibility of a boundary, we call it a manifold with boundary. It is called a *manifold with nonnull boundary* if we want to say that it has a boundary for certain.

For a manifold with nonnull boundary (such as a ball, arc, disk), we use Bd M to denote the boundary of M (set of points which have open neighborhoods homeomorphic to closed Euclidean half space) and Int M to denote the interior of M (set of points with open neighborhoods homeomorphic to Euclidean space).

When we call a manifold or geometric object *triangulated*, we mean that a particular triangulation is assigned to it. If no such assignment has been made but the object is isometric to some geometric complex, we say that the object is *polyhedral.* If the object is topologically equivalent to some geometric complex, we say that it is *triangulable* or *can be triangulated.* For example, in the plane, a polygon is regarded as triangulated and the closed disk bounded by the polygon is polyhedral. If a particular triangulation were assigned to the polygonal disk, it would

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then be triangulated. A round disk would be neither triangulated nor polyhedral, but it would be triangulable.

Since it is known ([3], [4], [8], [9]) that any 3-manifold with boundary is triangulable, it does not restrict those being studied to limit our consideration to those that are polyhedral. By using the polyhedral requirement we are able to specify that certain of our maps are piecewise linear and certain subsets are polyhedral.

A 2-manifold is *orientable* if it contains no Möbius band; a 3-manifold is *orientable* if it contains no solid Klein bottle. A solid Klein bottle is a set topologically equivalent to the cartesian product of a Möbius band and a segment.

A closed *polyhedral* object is called a polyhedron. We do not insist that it it be compact. A *polyhedral n-cell* is a polyhedron that can be given a triangulation compatible with its metric which is isomorphic to some rectilinear triangulation of the ordinary n-cell. A compact 1-dimensional polyhedron is called a polyhedral 1-complex.

We say that there is an *elementary geometric collapse* of a polyhedron P_1 onto a subpolyhedron P_2 if for some integer m there is a polyhedral m-cell B^m such that $P_1 = P_2 \cup B^m$ and $P_2 \cap B^m$ is a polyhedral $(n-1)$-cell on Bd B^m. We write $P_1 \searrow P_2$. We say that P collapses to Q if there is a finite sequence of elementary collapses such that

$$P = P_1 \searrow P_2 \searrow \cdots \searrow P_j = Q.$$

Suppose P is a compact polyhedron in a polyhedral n-manifold with boundary M^n. We say that N^n is *a regular neighborhood of* P in M^n if N^n is an n-manifold with boundary in M^n, N^n collapses to P, and P lies in an open subset of M^n in N^n. It is known ([7], [10], [12]) that if N_1^n, N_2^n are regular neighborhoods of P, then there is a piecewise linear homeomorphism of N_1^n onto N_2^n.

We call an i-dimensional polyhedron P^i an *i-spine* of a compact polyhedral n-manifold with boundary M^n if M^n collapses to P^i and P^i does not collapse onto any proper subpolyhedron.

A collection G of mutually exclusive compact sets in a metric space X is called an *upper semicontinuous collection* if X is the union of elements of G and for each open set U in X containing an element of G, the union of elements of G in U is open. The *decomposition space* X/G has as points the elements of G and as open sets collections of these "points" whose union is open in X. The *decomposition map* of X onto X/G is the one which sends the point $x \in X$ into the "point" $g_X \in X/G$ which contains it (when g_X is regarded as a set in X). The decomposition map may alternatively be called an *upper semicontinuous decomposition* or for brevity merely a decomposition. We speak of the decomposition of X as being *into* sets (elements of G) and *onto* X/G. The decomposition is *monotone* if each element of G is connected.

If A is a compact subset of X rather than an upper semicontinuous decomposition of X, we use X/A to denote X/G_A where G_A is the upper semicontinuous decomposition of X whose only nondegenerate element is A.

Closely related to the notion of a monotone upper semicontinuous decomposition is that of a monotone mapping. A continuous transformation is called a *map* or *mapping*. A map f of X is *monotone* if the point inverses are compact continua—that is, for each $y \in f(X)$, $f^{-1}(y)$ is compact and connected.

If f is a monotone map of a space X onto a space Y such that the point inverses form an upper semicontinuous collection G, consideration of the 1-1 continuous map of $[0, \infty)$ onto a circle shows that one cannot conclude that Y is homeomorphic to X/G. However, they are homeomorphic if the map f is compact. A mapping is *compact* if the inverses of compact sets are compact. See [11] for a discussion of relationships between decompositions and mappings.

A map f of a polyhedron P_1 into a polyhedron P_2 is *piecewise linear* if P_1, P_2 have triangulations T_1, T_2 compatible with the metrics of P_1, P_2 so that f takes each simplex of T_1 linearly *into* a simplex of P_2. In case P_1 is compact we can say that f takes each simplex of T_2 linearly *onto* a simplex of T_1 but for noncompact polyhedra, this is too restrictive. If $g(x)=1/x$ $(x \geqq 1)$ there is a piecewise linear map f of $[1, \infty)$ into $[0, 1]$ such that $|f(x)-g(x)| \leqq 1/2x$.

Throughout this paper we shall use M^3 to denote a *compact, connected, polyhedral* 3-*manifold with boundary*. Perhaps it has no boundary but if this is required, it is so specified.

2. **Boring holes in M^3.** Suppose H^3 is a polyhedral 3-cell in M^3 so that $H^3 \cap \text{Bd } M^3$ is the union of two disjoint disks D_0^2, D_1^2. If we form a new 3-manifold with boundary M_1^3 by removing $\text{Int } H^3 \cup \text{Int } D_0^2 \cup \text{Int } D_1^2$ from M^3, we say that H^3 is a *hole bored* in M^3, and D_0^2, D_1^2 are the *ends* of the hole. We say that M_1^3 *was obtained by boring a hole in* M^3.

It is known that if M^3 has a nonnull boundary, then it can be changed into a cube with handles (some of the handles will be twisted or nonoriented if M^3 is not orientable) by boring a finite number of holes in M^3. A convenient way to find the holes is to bore them about spanning arcs in the 1-skeleton of some triangulation of M^3. Theorem 2.2 is a variation of this result. In order to prove Theorem 2.2 we first use Theorem 2.1 to build a 2-spine Q^2 which perhaps looks like 3-pages of a book at some places. Casler proved a stronger version of Theorem 2.1 in [6] where he built a Q^2 with a standard structure. We do not need this standard structure so for completeneness we include an elementary proof of Theorem 2.1.

THEOREM 2.1. *If M^3 has a nonnull boundary, there is a spine Q^2 for M^3 such that if two disks in Q^2 intersect in an interior point of each, their intersection is* 2-*dimensional.*

Proof. Let T be a triangulation of M^3. Our intention is to change T to a cellular decomposition T' such that if two disks in the 2-skeleton of T' intersect in an interior point of each, their intersection is 2-dimensional.

Blunt each 3-simplex of T by removing 3-simplexes about each vertex as shown in Figure 2 on p. 17 of [1]. Each of these removed simplexes is similar to the original and has edges one-third as long. For each vertex of T, the union of the blunted ends containing the vertex is a 3-cell which is an element of the cellular subdivision T'.

Consider each of the blunted 3-simplexes. What is left of its edges are trimmed off as shown in Figure 2 on p. 17 of [1]. The union of the prismatic pieces about the middle third of each edge is a 3-cell which is an element of the subdivision T'.

The other 3-dimensional cells in T' are the blunted and trimmed 3-simplexes of T.

The spine Q^2 promised by Theorem 2.1 is obtained by considering the 3-cells of T' and ordering them $B_1^3, B_2^3, \dots, B_m^3$ so that each Bd B_i^3 contains a polyhedral 2-cell B_i^2 which it shares in common with either Bd M^3 or some Bd B_j $(j<i)$. Then

$$M^3 \searrow M^3 - \operatorname{Int} B_1^3 - \operatorname{Int} B_1^2 \searrow \cdots \searrow M^3 - \bigcup_{i=1}^{m} \operatorname{Int} B_i^3 - \bigcup_{i=1}^{m} \operatorname{Int} B_i^2 = Q_1^2.$$

Then Q^2 is obtained by collapsing Q_1^2 as much as possible. That M^3 is a regular neighborhood of this reduced Q^2 follows from the result ([7], [10], [12]) that if one polyhedron collapses to a second, each regular neighborhood of the larger polyhedron is a regular neighborhood of the smaller. Likely Q^1 is a 2-spine but its dimension might be less than 2.

THEOREM 2.2. *If M^3 has precisely two boundary components, it is possible to obtain a 3-manifold with boundary M_0^3 from M^3 by boring a finite number of holes in it such that for some polyhedral 2-manifold M^2 there is a piecewise linear homeomorphism of $M^2 \times [0, 1]$ onto M_0^3.*

Proof. We shall show that by boring holes in M^3, it can be changed to a polyhedral 3-manifold with boundary M_0^3 such that M_0^3 has two boundary components and M_0^9 collapses to a 2-spine M^2 which is a 2-manifold M^2. The truth of Theorem 2.2 then follows from the facts that M_0^3 is a regular neighborhood of M^2, any two regular neighborhoods of M^2 are piecewise linearly homeomorphic, and there is a regular neighborhood of M^2 piecewise linearly homeomorphic to $M^2 \times [0, 1]$.

It follows from Theorem 2.1 that M^3 has a 2-spine Q^2 such that if two disks in Q^2 meet in an interior point of each, their intersection is 2-dimensional. We suppose Q^2 is such a spine that lies in Int M^3. The 2-manifold M^2 promised by the preceding paragraph is obtained by altering Q^2.

If there is a 1-cell A_1 and three disks D_1^2, D_2^2, D_3^2 in Q^2 such that $D_i^2 \cap D_j^2 = A^1$ $(i \neq j)$, some one of the disks is not a subset of the part of Q^2 irreducible with respect to separating the two boundary components from each other in M^3. Holes can be bored in M^3 and through this disk so as to obtain a 3-manifold with two boundary components which collapses onto a proper subpolyhedron of Q^2.

Let M_1^3 be a 3-manifold with two boundary components obtained by boring holes in M^3 such that M_1^3 contains a spine $Q_1^2 \subset Q^2$ such that no further boring of holes in M_1^3 results in a 3-manifold with two boundary components and a spine which is a proper subset of Q_1^2. Note that Q_1^2 is the union of a 2-manifold M_1^2 and a 1-complex. By adjusting this 1-complex we may suppose that it is the union of a finite number of disjoint arcs $A_1^1, A_2^1, \ldots, A_m^1$ such that each $A_i^1 \cap M_1^2 = \text{Bd } A_i^1$.

Boring holes in M_1^3 with the A_i^1's as centers of the holes produces a 3-manifold with boundary M_0^3 such that M_0^3 has two boundary components and a 2-manifold M^2 as a spine where M^2 is obtained from M_1^2 by replacing the A_i^1's and disks on M_1^2 about their ends by annuli.

3. **Decomposing M_3 into 2-manifolds and pinched 2-manifolds.** A *pinched manifold* results from identifying two points of the same manifold. If two objects are joined at a point, their union is called the *wedge* of the two objects. The wedge of two tangent round 2-spheres is an example of a pinched 2-manifold. If a cube with handles has one of its handles squeezed to a point, the boundary of the resulting object may be an example of a pinched 2-manifold.

Suppose B^2 is a boundary component of M^3. It is known (see for example Lemma 1 of [8]) that there is a piecewise linear homeomorphism h of $B^2 \times [0, 1]$ into M^3 so that for each $x \in B^2$,

$$h(x \times 0) = x \quad \text{and} \quad h(B^2 \times (0, 1]) \subset \text{Int } M^3.$$

We call $h(B^2 \times [0, 1])$ a *polyhedral cartesian product neighborhood of* B^2.

In the following three theorems we show that certain 3-manifolds can be decomposed into 2-manifolds and pinched 2-manifolds. A map is *open* if the images of open sets are open.

THEOREM 3.1. *Suppose M^3 has precisely two boundary components B_0^2, B_1^2. Then there is a monotone piecewise linear open map f of M^3 onto* $[0, 1]$ *such that* $f^{-1}(0) = B_0^2, f^{-1}(1) = B_1^2$, *and for each* $t \in [0, 1], f^{-1}(t)$ *is either a 2-manifold or a pinched 2-manifold. If* $[a, b] \subset [0, 1]$ *and for each* $t \in [a, b], f^{-1}(t)$ *is a 2-manifold, there is a piecewise linear homeomorphism of* $f^{-1}[a, b]$ *onto* $f^{-1}(a) \times [a, b]$ *that takes* $f^{-1}(t)$ *onto* $f^{-1}(a) \times t$.

Proof. Let h_0 be a piecewise linear homeomorphism of $B_0^2 \times [0, 1/3]$ onto a cartesian product neighborhood of B_0^2 in M^3 so that $h_0(x \times 0) = x$ and let h_1 be a piecewise linear homeomorphism of $B_1^2 \times [2/3, 1]$ onto a cartesian product neighborhood of B_1^2 in M^3 so that $h_1(x \times 1) = x$. We suppose these two cartesian product neighborhoods do not intersect each other. Then

$$M^3 - h_0(B_0^2 \times [0, 1/3)) - h_1(B_1^2 \times (2/3, 1]) = M_1^3$$

is a 3-manifold with precisely two boundary components $h_0(B_0^2 \times 1/3)$, $h_1(B_1^2 \times 2/3)$.

It follows from Theorem 2.2 that it is possible to bore a finite number of mutually exclusive polyhedral holes $H_1^3, H_2^3, \ldots, H_n^3$ in M_1^3 such that each of these holes has both ends on the same boundary component of M_1^3 and for some 2-manifold M^2, there is a piecewise linear homeomorphism h of $M^2 \times [1/3, 2/3]$ onto the closure of $(M_1^3 - \bigcup H_i^3)$. We suppose $h(M^2 \times 1/3)$, $h(M^2 \times 2/3)$ are the altered $h_0(B_0^2 \times 1/3)$, $h_1(B_1^2 \times 2/3)$ respectively and define

$$f(h(M^2 \times t)) = t, \qquad t \in [1/3, 2/3].$$

Let $H_1^3, H_2^3, \ldots, H_m^3$ be the holes bored in M_1^3 from $h_0(B_0^2 \times 1/3)$. For each H_i^3, let C_i^3, $C_i^{3\prime}$ be caps over the ends of H_i^3 so that each C_i^3 is a polyhedral 3-cell in $h_0(B_0^2 \times [0, 1/3])$ such that $C_i^3 \cap h_0(B_0^2 \times 1/3)$ is an end of H_i^3, Bd $C_i^3 \cap h_0(B_0^2 \times 1/6)$ is a point and for each $t \in (1/6, 1/3)$, Bd $C_i^3 \cap h_0(B_0^2 \times t)$ is a simple closed curve; the $C_i^{3\prime}$ are similarly defined. We suppose that no two of the caps intersect each other.

If $t \in [0, 1/3)$ and $p \in h_0(B_0^2 \times t) - \bigcup \text{Int } C_i^3 - \bigcup \text{Int } C_i^{3\prime}$, we define $f(p) = t$.

For each $i = 1, 2, \ldots, m$, consider the polyhedral 3-cell $H_i^3 \cup C_i^3 \cup C_i^{3\prime}$. We note that f has already been defined on its boundary. Triangulate this boundary so that f is linear with respect to the triangulation. Then triangulate $H_i^3 \cup C_i^3 \cup C_i^{3\prime}$ by coning from an interior point p_i.

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ be different numbers in $(1/6, 1/3)$. Define $f(p_i) = \varepsilon_i$ and extend f linearly to the simplexes of the conical triangulation of $H_i^3 \cup C_i^3 \cup C_i^{3\prime}$. Then $(H_i^3 \cup C_i^3 \cup C_i^{3\prime}) \cap f^{-1}(t)$ is the union of two points if $t = 1/6$, the union of two disjoint disks (cones) if $t \in (1/6, \varepsilon_i)$, the union of two cones with a common vertex p_i if $t = \varepsilon_i$, and an annulus if $t \in (\varepsilon_i\ 1/3]$. It is to be noted that f is open at p_i.

Note that for some neighborhood $(\varepsilon_i - \delta, \varepsilon_i + \delta)$ of ε_i, there is a piecewise linear projection r_i of $f^{-1}[\varepsilon_i - \delta, \varepsilon_i + \delta]$ onto $f^{-1}(\varepsilon_i)$ such that for $t \in [\varepsilon_i - \delta, \varepsilon_i)$, r_i restricted to $f^{-1}(t)$ is one-to-one except onto p_i and the inverse of p_i is a pair of points while if $t \in (\varepsilon_i, \varepsilon_i + \delta]$, r_i restricted to $f^{-1}(t)$ is one-to-one except onto p_i and the inverse of p_i is a simple closed curve.

In a similar fashion we define f on $h_1(B_1^2 \times [2/3, 1])$ and the holes with ends on $h_1(B_1^2 \times 2/3)$.

Definitions. A *dendron* is a compact locally connected metric continuum which contains no simple closed curve. A point p is of *order* $\leqq n$ if for each neighborhood U of p there is an open set V with $p \in V \subset U$ such that the boundary (frontier) of V has at most n points. A point is of *order* n if it is of order $\leqq n$ but not of order $\leqq n-1$. A point of order 1 is an *end point* of the dendron and a point which is not of order $\leqq 2$ is a *branch point*. A polyhdral dendron (sometimes called a *tree*) can have at most a finite number of branch and end points. A dendron with precisely three end points (and hence one branch point) is called a *triod*.

The following is a variation of Theorem 3.1.

Theorem 3.2. *If M^3 has precisely three boundary components, there is a piecewise linear monotone map f of M^3 onto a polyhedral triod T^1 such that each boundary component of M^3 is an inverse of an end point of T^1 and for each $t \in T^1$, $f^{-1}(t)$ is either a 2-manifold or a pinched 2-manifold.*

Proof. Let B_1^2, B_2^2 be two of the boundary components of M^3; h_1 be a piecewise linear homeomorphism of $B_1^2 \times [0, 1]$ onto a cartesian product neighborhood of

B_1^2 in M^3 such that for each $x \in B_1^2$, $h_1(B_1^2 \times 0) = x$; and h_2 be a piecewise linear homeomorphism of $B_2^2 \times [0, 1]$ onto a cartesian product neighborhood of B_2^2 in M^3 such that for each $x \in B_2^2$, $h_2(x \times 0) = x$. We suppose these two cartesian product neighborhoods are disjoint.

Let H^3 be a hole in $M^3 - h_1(B_1^2 \times [0, 1)) - h_2(B_2^2 \times [0, 1))$ with one end on $h_1(B_1^2 \times 1)$ and the other on $h_2(B_2^2 \times 1)$. It follows by the methods of the proof of Theorem 3.1 that there is a monotone piecewise linear map f_1 of

$$H^3 \cup h_1(B_1^2 \times [0, 1]) \cup h_2(B_2^2 \times [0, 1])$$

into a polyhedral triod T_1^1 such that the inverse of the branch point of T_1^1 is the wedge of two 2-manifolds, the inverse of each other point of T_1^1 is a 2-manifold. The inverses of the end points of the triod are $h_1(B_1^2 \times 0)$, $h_2(B_2^2 \times 0)$, and the boundary component of $H^3 \cup h_1(B_1^2 \times [0, 1]) \cup h_1(B_2^2 \times [0, 1])$ intersecting Bd H^3.

It follows from Theorem 3.1 that the piecewise linear map f_1 of

$$H^3 \cup h_1(B_1^2 \times [0, 1]) \cup h_2(B_2^2 \times [0, 1])$$

onto T_1^1 can be extended to a piecewise linear map f of M^3 onto a larger triod T^1 satisfying the conditions of Theorem 3.2. The map f we define is open except on the inverse of the branch point of the triod.

In a certain sense, we proved Theorem 3.2 by combining two boundary components of M^3. Repetitions of this procedure permits us to prove the following extension of Theorem 3.2.

THEOREM 3.3. *If M^3 has more than one boundary component, there is a piecewise linear monotone map f of M^3 onto a tree T^1 such that each point of T^1 is of order ≤ 3, the boundary components of M^3 are the inverses of end points of T^1, and for each $t \in T^1$, $f^{-1}(t)$ is either a 2-manifold or a pinched 2-manifold.*

Theorem 7.2 of §7 gives results about the mapping of M^3 onto a dendron if M^3 has no boundary.

4. **Decomposition maps onto 2-spheres.** In the last section we decomposed M^3 into 2-manifolds and pinched 2-manifolds by letting these be the point inverses in a map onto a dendron. In §6 we shall map M^3 onto a punctured cube. This is done by considering the point inverses in the map onto a dendron and decomposing M^3 by decomposing each of these inverses. In this section we consider useful decompositions of 2-manifolds and pinched 2-manifolds.

Consider a simplex Δ which is the join of two simplexes Δ_1, Δ_2. Then $\Delta = \Delta_1 \circ \Delta_2$ is the union of segments from Δ_1 to Δ_2. For each $t \in [0, 1]$, let A_t be the set of all points such that for some $p_1 \in \Delta_1$, $p_2 \in \Delta_2$, p divides the segment from p_1 to p_2 in the ratio t to $1-t$. We denote p by (p_1, p_2, t) and note that

$$A_t = \{(p_1, p_2, t) \mid p_1 \in \Delta_1, p_2 \in \Delta_2\}.$$

Then $A_0 = \Delta_1$, $A_1 = \Delta_2$, and the midsection $A_{1/2}$ is homeomorphic to $\Delta_1 \times \Delta_2$. We use A_B to denote $\bigcup_{t \in B} A_t$.

Let Δ/Δ_1 be the cell obtained from Δ by shrinking Δ_1 to a point. To give Δ/Δ_1 a structure we regard it as the union of $A_{[1/2,1]}$ and the cone C over $A_{1/2}$ from a point v_0 where $A_{[1/2,1]} \cap C = A_{1/2}$.

The natural decomposition map f from Δ to Δ/Δ_1 is the identity on $A_{[1/2,1]}$ and for $t \in [0, 1/2]$, $f(p_1, p_2, t)$ is the point that divides the segment from v_0 to $(p_1, p_2, 1/2)$ in the ratio $2t$ to $1-2t$.

We note that although Δ is a simplex and $f(\Delta) = \Delta/\Delta_1$ is a piecewise linear cell, the map f is not piecewise linear.

If L is a subcomplex of complex K, K/L results by regarding L as a point. To give K/L a cellular structure, we let T' be the first barycentric subdivision of the triangulation of K and note that if Δ is a simplex in T', then either Δ misses L, lies in L, or intersects it in a proper face Δ_L. The natural decomposition map f is such that $f(L)$ is a vertex v_0 of K/L; f is the identity on each simplex of T' that misses L; for each simplex Δ of T' whose intersection with L is a proper face Δ_L of Δ, f is the natural decomposition map from Δ to Δ/Δ_L.

The following result is well known.

THEOREM 4.1. *If M^2 is a compact connected polyhedral 2-manifold, there is a connected polyhedral 1-complex C^1 in M^2 such that M^2/C^1 is a polyhedral 2-sphere.*

To obtain C^1, triangulate M^2 and remove an open 2-simplex $\operatorname{Int} \Delta^2$ from M^2. Then C^1 is obtained by collapsing $M^2 - \operatorname{Int} \Delta^2$.

The following two results may be proved in a similar fashion.

THEOREM 4.2. *If N^2 is a compact connected triangulated pinched 2-manifold with no separating point, there is a connected polyhedral 1-complex C^1 in the 1-skeleton of the triangulation such that N^2/C^1 is a polyhedral 2-sphere and C^1 is of order 2 at the pinch point.*

We note that C^1 abuts on the pinch point from both sides in N^2.

THEOREM 4.3. *If N^2 is a compact connected triangulated pinched 2-manifold which is separated by its pinch point, there is a connected polyhedral 1-complex C^1 in the 1-skeleton of the triangulation such that C^1 abuts on the pinch point from both sides and is of order 2 there while N^2/C^1 is the wedge of two polyhedral 2-spheres.*

5. **Changing pseudo spines.** A *pseudo spine* for a compact connected polyhedral 2-manifold is a connected polyhedron C in M^2 such that M^2/C is topologically a 2-sphere. To get C one might remove an open 2-simplex from M^2 and collapse the remainder. Note that the pseudo spine differs from a spine for M^2 in that an open simplex was removed before the collapsing. Also, the collapsing was not necessarily continued as much as possible. For example, a 1-simplex on S^2 serves as a pseudo spine for it. To emphasize that the pseudo spines we shall be using are 1-dimensional polyhedra, we may call them pseudo 1-spines.

If M^2 is a compact, connected, polyhedral 2-manifold, $M^2 \times [0, 1]$ is its cartesian product with an interval, and $C_0^1 \times 0$, $C_1^1 \times 1$ are pseudo 1-spines of $M^2 \times 0$, $M^2 \times 1$, we shall be interested in finding pseudo 1-spines for the intermediate layers.

THEOREM 5.1. *Suppose M^2 is a compact connected polyhedral 2-manifold and C_0^1, C_1^1 are pseudo 1-spines of M^2. Then C_0^1, C_1^1 can be extended to a 1-parameter family of pseudo 1-spines C_t^1 $(0 \leqq t \leqq 1)$ of M^2 such that there is a monotone map f of $M^2 \times [0, 1]$ onto $S^2 \times [0, 1]$ with $f(M^2 \times t)$ being equal to $S^2 \times t$ and $C_t^1 \times t$ being the only nondegenerate inverse in $M^2 \times t$.*

Proof. Let $D_1^2, D_2^2, \ldots$ be a sequence of polyhedral disks in M^2 such that $D_i^2 \subset \text{Int } D_{i+1}^2$ and $\bigcup D_i^2 = M^2 - C_0^1$ and $E_1^2, E_2^2, \ldots$ be such a sequence of disks whose union is $M^2 - C_1^1$.

Since E_1^2 is a disk which lies on the interior of a larger disk in M^2 with the larger disk not being a subset of $M^2 - \text{Int } D_1^2$, there is a piecewise linear isotopy fixed except on this larger disk that pushes $M^2 - \text{Int } D_1^2$ off E_1^2. Denote this isotopy by H_t $(0 \leqq t \leqq 1/4)$ where $H_0 = \text{identity}$ and $E_1^2 \cap H_{1/4}(M^2 - \text{Int } D_1^2) = 0$. Then $H_{1/4}(D_1^2) - \text{Int } E_1^2$ is an annulus and H_t $(0 \leqq t \leqq 1/4)$ can be extended to a piecewise linear isotopy H_t $(0 \leqq t \leqq 1/2)$ such that $H_{1/2}(D_1^2) = E_1^2$.

The isotopy H_t $(0 \leqq t \leqq 1/2)$ can be extended to a piecewise linear isotopy H_t $(0 \leqq t \leqq 3/4)$ so that $H_{3/4}(D_2^2) = E_2^2$ and for $x \in D_1^2$ and $1/2 \leqq s \leqq 3/4$, $H_s(x) = H_{1/2}(x)$.

Similarly, we extend H_t to H_t $(0 \leqq t < 1)$. In general, H_t is such that $H_{t_i}(D_i) = E$ for $t_i = (1 - 1/2^i)$ and $H_s = H_{t_i}$ on D_i for $s \geqq (1 - 1/2^i)$.

For $0 \leqq t < 1$, $C_t^1 = H_t(C_0^1)$.

Let f_0 be a decomposition map of M^2 onto S^2 such that C_0^1 is the only nondegenerate inverse. The map f of $M^2 \times [0, 1)$ is defined by $f(x \times t) = f_0 H_t^{-1}(x) \times t$. It is extended to $M^2 \times 1$ by continuity.

QUESTIONS. It would appear that Theorem 5.1 is a special case of some more general theorem. If f is a map of a compact continuum X onto $[0, 1]$ such that the point inverses are all homeomorphic to each other, under what condition can it be concluded that X is homeomorphic to $f^{-1}(0) \times [0, 1]$? If g is a monotone map of $Y \times [0, 1]$ onto Z such that the $g(Y \times t)$'s are homeomorphic but no two intersect, under what condition is Z homeomorphic to $g(Y \times 0) \times [0, 1]$? Is it enough that each $f(Y \times t)$ be a 2-sphere? What if each $Y \times t$ contain at most one nondegenerate point inverse? Something needs to be imposed because if P is a pseudo arc, P' is a nondegenerate proper subcontinuum of $P \times 0$, then $(P \times [0, 1])/P'$ is not homeomorphic to $P \times [0, 1]$ since it contains a triod. However it is known [2] that $(P \times 0)/P'$ is homeomorphic to P.

It is to be noted that the map f is not piecewise linear and is especially bad at $t = 1$. Can we find such an f that has the nice properties of a natural decomposition map?

6. **Decomposing 3-manifolds with boundaries.** Suppose $B_0^3, B_1^3, \ldots, B_n^3$ are

polyhedral 3-cells such that $B_i^3 \subset \operatorname{Int} B_0^3$ for $i=1, 2, \dots, n$ and $B_i^3 \cap B_j^3 = 0$ for $0 < i < j \leqq n$. Then $B_0^3 - \bigcup_{i=1}^n \operatorname{Int} B_i^3$ is a *polyhedral punctured cube.*

If the boundary components of M^3 are already nicely decomposed with the decomposition space of each being a 2-sphere, we shall be interested in extending the decomposition to M^3 so that the decomposition space is a punctured cube.

THEOREM 6.1. *Suppose M^3 has precisely two boundary components B_0^2, B_1^2 with pseudo 1-spines C_0^1, C_1^1. Then there is a monotone map g of M^3 onto $S^2 \times [0, 1]$ such that $g^{-1}(S^2 \times 0) = B_0^2$, $g^{-1}(S^2 \times 1) = B_1^2$, each $g^{-1}(S^2 \times t)$ is either a polyhedral 2-manifold or polyhedral pinched 2-manifold, C_0^1 and C_1^1 are point inverses, each nondegenerate point inverse is a polyhedral 1-complex, and the image of the union of the nondegenerate point inverses is an arc from $S^2 \times 0$ to $S^2 \times 1$ that intersects each $S^2 \times t$ only once.*

Proof. Let f be a monotone piecewise linear map of M^3 onto [0, 1] as described in Theorem 3.1 such that $f^{-1}(0) = B_0^2$, $f^{-1}(1) = B_1^2$, and each inverse is either a 2-manifold or a pinched 2-manifold. We plan to define g so that $gf^{-1}(t) = S^2 \times t$. To get such a g we need to get a 1-complex C_t^1 on each $f^{-1}(t)$ so that $f^{-1}(t)/C_t^1$ is homeomorphic to S^2. We shall first describe the C_t^1's on the pinched inverses, then on the inverses near these pinched inverses, and finally use Theorem 5.1 to fill in the gaps.

Since f is piecewise linear, there are only a finite number of pinched 2-manifolds among the inverses. Let $0 = t_1 < t_2 < \cdots < t_{3n+2} = 1$ be such that the pinched inverses are the $f^{-1}(t_{3i})$'s ($i = 1, 2, \dots, n$). For each $i = 1, 2, \dots, n$, let $C_{t_{3i}}^1$ be a polyhedral 1-complex in $f^{-1}(t_{3i})$ such as guaranteed by Theorem 4.2 so that $f^{-1}(t_{3i})/C_{t_{3i}}^1$ is a 2-sphere.

In the proof of Theorem 3.1 we analyzed the nature of the map f in the neighborhood of the pinched inverses. There is a projection of the $f^{-1}(t)$'s near $f^{-1}(t_{3i})$ onto $f^{-1}(t_{3i})$ so that the projection is one-to-one except onto the pinched point and the inverse of the pinched point under the projection is either a pair of points or a polyhedral simple closed curve. For $t \in [t_{3i-1}, t_{3i+1}]$ let C_t^1 be the inverse of $C_{t_{3i}}^1$ under the projection of $f^{-1}(t)$ onto $f^{-1}(t_{3i})$. There is a map g_i of $f^{-1}[t_{3i-1}, t_{3i+1}]$ onto $S^2 \times [t_{3i-1}, t_{3i+1}]$ such that $gf^{-1}(t) = S^2 \times t$ and C_t^1 is the only nondegenerate inverse in $g^{-1}(S^2 \times t)$.

Although there is a piecewise linear homeomorphism of $f^{-1}[t_{3j-2}, t_{3j-1}]$ for $j = 1, 2, \dots, n+1$ onto $f^{-1}(t_{3j-2}) \times [t_{3j-2}, t_{3j-1}]$, there is no assurance that $C_{t_{3j-2}}^1$ and $C_{t_{3j-1}}^1$ line up. However Theorem 5.1 shows that polyhedral C_t^1's can be selected for $t \in (t_{3j-2}, t_{3j-1})$ so as to insure that there is a level preserving monotone map of $f^{-1}[t_{3j-2}, t_{3j-1}]$ onto $S^2 \times [t_{3j-2}, t_{3j-1}]$ so that the only nondegenerate inverses are the C_t^1's.

The map g is obtained by patching together the various maps of the $f^{-1}[t_{3i-1}, t_{3i+1}]$ onto the $S^2 \times [t_{3i-1}, t_{3i+1}]$ for $i = 1, 2, \dots, n$ and the maps of the $f^{-1}[t_{3j-2}, t_{3j-1}]$ onto the $S^2 \times [t_{3j-2}, t_{3j-1}]$ for $j = 1, 2, \dots, n+1$.

THEOREM 6.2. *If M^3 has only one boundary component B^2 and C^1 is a pseudo 1-spine of B^2, then there is a monotone map g of M^3 onto a polyhedral 3-cell B^3 such that $g^{-1}(\text{Bd } B^3)=B^2$, each nondegenerate inverse is a polyhedral 1-complex, C^1 is the only such inverse on B^2, and the image under g of the union of the nondegenerate inverses is a polyhedral arc in B^3.*

Proof. A polyhedral open ball is removed from M^3 to change it to a 3-manifold with two boundary components. A polygonal arc is selected to serve as a pseudo 1-spine for the new boundary component. Theorem 6.1 assures us that there is a suitable map of the remainder onto $S^2\times[0, 1]$. This map may be extended to all of M^3 to get the required map onto B^3.

THEOREM 6.3. *Suppose M^3 has boundary components $B_1^2, B_2^2, \ldots, B_n^2$ $(n\geqq 2)$ with polyhedral pseudo 1-spines $C_1^1, C_2^1, \ldots, C_n^1$. Then there is a monotone map g of M^3 onto a polyhedral punctured cube K^3 with boundary components $E_1^2, E_2^2, \ldots, E_n^2$ such that $g^{-1}(E_i^2)=B_i^2$, C_i^1 is the only nondegenerate inverse on B_i^2, each nondegenerate inverse is a polyhedral 1-complex, and the image under g of the union of the nondegenerate inverses is a tree in K^3 each of whose ends is on* Bd K^3.

Proof. Let f be a piecewise linear map such as guaranteed by Theorem 3.3 and described in the proofs of Theorems 3.1 and 3.2 such that $f(M^3)$ is a tree T^1, each branch point of T^1 is of order 3, the inverses of the end points of T^1 are the boundary components of M^3, and for each $x\in T^1$, $f^{-1}(x)$ is either a 2-manifold or a pinched 2-manifold.

Let $f^{-1}(x_1), f^{-1}(x_2), \ldots, f^{-1}(x_{n-2})$ be the pinched 2-manifolds among the inverses which are separated by their pinch points. Each $f^{-1}(x_i)$ separates M^3 into three parts. Assign a 1-complex $C_{x_i}^1$ to $f^{-1}(x_i)$ which satisfies the conditions of Theorem 4.3.

Let $T_1^1, T_2^1, \ldots, T_{n-2}^1$ be mutually exclusive trees in T^1 about the x_i's such that the branch point of T_i^1 is the only point of it that has an inverse which is pinched and no end point of T^1 is in any T_i^1.

For each $x\in T_i^1$, there is a natural projection of $f^{-1}(x)$ into $f^{-1}(x_i)$. From two sides, the projection is one-to-one onto manifolds in $f^{-1}(x_i)$ and from the third side the projection is one-to-one onto $f^{-1}(x_i)$ except onto the pinch point and the inverse of the pinch point is a polyhedral simple closed curve.

For the $f^{-1}(x)$'s that project one-to-one onto a 2-manifold in $f^{-1}(x_i)$, we let C_x^1 be the inverse under the projection of the part of $C_{x_i}^1$ in this 2-manifold.

For the $f^{-1}(x)$'s that project onto all of $f^{-1}(x_i)$, we let C_x^1 be the inverse of $C_{x_i}^1$ under the projection with an open arc removed from the inverse of the pinch point under the projection so that $f^{-1}(x)/C_x^1$ is a 2-sphere rather than the wedge of two 2-spheres. There is a monotone map g_1 of $f^{-1}(T_i^1)$ onto a polyhedral punctured cube K_i^3 with three boundary components such that C_x^1 is the only nondegenerate inverse intersecting $f^{-1}(x)$.

Consider the components of the part of T^1 for which C^1's have not been assigned to the inverses of its points. There are only a finite number of such components and the closure of each is an arc A. Already C^1's have been assigned to the inverses of the ends of A. Theorem 6.1 provides C^1's to the inverses of other points of A as well as a monotone map of $f^{-1}(A)$ into a hollow ball. The map g promised by Theorem 6.3 follows by patching together the monotone maps g_i on the $f^{-1}(T_i^1)$'s and those on the A's.

7. **Extending monotone mappings.** In this section we suppose M^3 is without boundary and X is a closed proper subset of M^3 such that if U is a connected open subset of M^3, then either $U-(X\cap U)$ is connected or X intersects the boundary (frontier) of U.

The principal result (Theorem 7.3) of this section is that there is a monotone map of M^3 onto S^3 that has the components of X as point inverses. Before proving this result, we prove some theorems about X.

THEOREM 7.1. *There is a decreasing sequence $M_1^3, M_2^3, \ldots$ such that $X=M_1^3 \cap M_2^3 \cap \cdots$ where each M_i^3 is a 3-manifold with boundary which is a polyhedron in M^3, $M_{i+1}^3 \subset M_i^3$, and each component of M_i^3 has a connected boundary.*

Proof. Suppose M_i^3 has been constructed. We prove the theorem by showing how to get an M_{i+1}^3 in any open subset V of Int M_i^3 which contains X.

Let T be a triangulation of M^3 of mesh less than one-half the distance from X to M^3-V and T'' be the second barycentric subdivision of T. Let N^3 be the union of the closed simplexes of T which intersect X and N_1^3 be the union of the closed simplexes of T'' which intersect N^3. Then N_1^3 is a regular neighborhood of N^3 and is hence a polyhedral 3-manifold with boundary ([7], [10], [12]). If C^3 is a component of N_1^3, it may have several boundary components, but boring holes in C^3 which miss X reduces the number of boundary components to one. Hence M_{i+1}^3 is obtained from N_1^3 by boring holes.

The following result follows from Theorem 7.1 and repeated applications of Theorem 3.3.

THEOREM 7.2. *There is a dendron D^1 and a monotone map f of M^3 onto D^1 such that each point of D^1 is of order $\leqq 3$, D^1 is locally polyhedral mod its set of end points, the components of X are the inverses of the endpoints of D^1, the inverse of each non end point is either a polyhedral 2-manifold or a polyhedral pinched 2-manifold, and f is locally piecewise linear mod X.*

THEOREM 7.3. *There is a monotone map g of M^3 onto S^3 such that each component of X is a point inverse, $g(X)$ lies on a straight segment, and each nondegenerate point inverse outside X is a polyhedral 1-complex. In fact, there are also a dendron D^1 in S^3 and a monotone map f of M^3 onto D^1 such that D^1 is locally polyhedral mod its set of end points, each branch point of D^1 is of order 3, X is the inverse under f of the set of end points of D^1, $f=g$ on X and on each nondegenerate point inverse under g,* *and if x is a non end point of D^1, $f^{-1}(x)$ is either a polyhedral 2-manifold or pinched 2-manifold, $f^{-1}(x)$ contains at most one nondegenerate point inverse under g, and $gf^{-1}(x)$ is either a 2-sphere or the wedge of two 2-spheres.*

Proof. The dendron D^1 and the map f are as given in Theorem 7.2, (except that the embedding of D^1 in S^3 is yet to be defined).

Let $T_1^1, T_2^1, \ldots$ be a monotone increasing sequence of trees in D^1 such that each point of order $\leqq 2$ of T_i^1 is a point of order 2 of T_{i+1}^1, each inverse under f of an end point of T_i^1 is a 2-manifold, and $\bigcup T_i^1$ contains each non end point of D^1.

By Theorem 6.3 there is a monotone map g_1 of $f^{-1}(T_1^1)$ onto a punctured cube K_1^3 such that each nondegenerate point inverse under g_1 is a polyhedral 1-complex, the image of the set of nondegenerate point inverses is a tree $T_{1'}^1$, and for each $x \in T_1^1$, $f^{-1}(x)$ contains exactly one such 1-complex, and $g_1 f^{-1}(x)$ is either a 2-sphere or the wedge of 2-spheres.

Let A be a straight segment in S^3. We suppose that K^3 lies in S^3, each component of $S^3 - K^3$ is of diameter less than $\frac{1}{2}$ and intersects A, $T_{1'}^1 = T_1^1$, $f = g$ on $f^{-1}(T_1^1)$.

Let $T_{2,i}^1$ $(i = 1, 2, \ldots, m)$ be the closures of the components of $T_2^1 - T_1^1$. It follows from Theorem 6.3 that there is a monotone map $g_{2,i}$ of $f^{-1}(T_{2,i}^1)$ onto a punctured cube $K_{2,i}^3$ such that each nondegenerate point inverse under $g_{2,i}$ is a polyhedral 1-complex, the image of the union of these 1-complexes is a tree $T_{2,i'}^1$, and for each $x \in T_{2,i}^1$, $f^{-1}(x)$ contains exactly one such 1-complex, while $g_{2,i} f^{-1}(x)$ is either a 2-sphere or the wedge of two 2-spheres.

We suppose that $K_{2,i}^3$ lies in S^3 so that only one component of $S^3 - K_{2,i}^3$ is of diameter more than 1/4, each component of $S^3 - K_{2,i}^3$ intersects A, and for $x_i = T_1^1 \cap T_{2,i}^1$, g_1, $g_{2,i}$ agree on $f^{-1}(x_i)$ with $g_1 f^{-1}(x_i) = g_{2,i} f^{-1}(x_i) = K_1^3 \cap K_{2,i}^3$, $T_{2,i'}^1 = T_{2,i}^1$, and $f = g$ on $f^{-1}(T_2^1)$.

For simplicity we use g to designate g_1 and the $g_{2,i}$'s. The map g is extended to $T_3^1, T_4^1, \ldots$ so that each component of $S^3 - gf^{-1}(T_i^1)$ is of mesh less than $1/2^i$ and intersects A. We place D^1 so that $f = g$ on $f^{-1}(D^1)$. Then g is extended to M^3 so that if y_0 is an end point of D^1 and y_i is the end point of T_i^1 toward y_0, $gf^{-1}(y_0)$ is the intersection of the small balls in S^3 bounded by the $gf^{-1}(y_i)$'s.

The unicoherence of S^3 permits us to weaken the restrictions on X (where such a weakening would not be possible for such a 3-manifold as $S^2 \times S^1$). Hence, by ignoring details about f we may state Theorem 7.3 for S^3 as follows.

THEOREM 7.4. *If Y is a closed subset of S^3 no component of which separates S^3, there is a monotone map of S^3 onto itself such that the components of Y are point inverses, each nondegenerate point inverse not in Y is a polyhedral 1-complex, and the image of the union of Y and these nondegenerate point inverses is a dendron.*

We can deal with noncompact 3-manifolds since each connected noncompact 3-manifold is the union of a monotone increasing sequence of connected compact

3-manifolds with boundaries. For example, for Euclidean 3-space E^3 we have the following result.

THEOREM 7.5. *Suppose Y is a closed subset of E^3 such that each component of Y is compact but does not separate E^3. Then there is a compact monotone map of E^3 onto itself such that the components of Y are point inverses, and nondegenerate point inverses not in Y are polyhedral* 1*-complexes. The image of the union of Y and the nondegenerate point inverses is a closed set whose* 1*-point compactification is a dendron with a closed set of end points.*

QUESTIONS. In [11] Whyburn raises the question as to whether or not each monotone map of E^3 onto itself need be compact. Could it be shown to be compact in case each point inverse is either a 1-complex or a component of Y? Would it help to know that the image of the union of the nondegenerate point inverses lies in a closed set whose 1-point compactification is a dendron?

It is to be noted that the maps g of Theorems 7.4, 7.5 were not necessarily cellular since some of the 1-complexes which were point inverses might contain cycles. Could one obtain a cellular g if the components of Y were pointlike? In particular, what happens if Y is the union of the tame arcs in the dogbone decomposition?

DEFINITIONS. Euclidean n-space ($n > 1$) has only *one end.* An *end* of a connected, locally compact, noncompact metric space S may be defined as an equivalence class of sequences $U_1, U_2, \ldots$ where U_i is a nonnull connected open subset of X with a compact boundary (frontier),

$$\overline{U_{i+1}} \subset U_i, \quad \text{and} \quad \bigcap U_i = 0.$$

The sequence $U_1, U_2, \ldots$ is in the same equivalence class as $V_1, V_2, \ldots$ if and only if the V_i's satisfy similar conditions, each U_i contains a V_j and each V_i contains a U_j. Hence, *S has one end* if for each compact set C, the closure of $S-C$ has precisely one noncompact component.

For other noncompact 3-manifolds, our methods give the following result.

THEOREM 7.6. *Suppose W^3 is a noncompact, connected, polyhedral,* 3*-manifold without boundary and Y is a closed subset of W^3 such that each component of Y is compact and if U is a connected open subset of W^3, either Y intersects the boundary of U or $U-(Y \cap U)$ is connected. Then there is a straight segment in S^3, a dendron D^1 which is locally polyhedral* mod *its set of end points and which intersects A in its set of end points, a closed subset C^0 of the set of end points of D^1, and a monotone map g of W^3 onto S^3-C^0 such that the components of Y are point inverses of the end points of D^1 not in C^0, the nondegenerate point inverses not in Y are polyhedral* 1*-complexes in W^3, and the image of the union of Y and the nondegenerate point inverses is D^1-C^0.*

The points C^0 correspond to the ends of W^3.

8. **Monotone maps that cannot be extended.** It would be interesting to know which monotone decompositions of a portion of a 3-manifold can be extended to the whole 3-manifold so that the decomposition space is a 3-manifold.

Suppose C_1, C_2 are two linking round circles in a simplex of S^3 and G is a decomposition of $C_1 \cup C_2$ whose only nondegenerate element is C_2. Is there a monotone upper semicontinuous decomposition of S^3 onto itself such that each element of G is an element of this decomposition? Is there a monotone map g of S^3 **onto** itself that is a homeomorphism on C_1 and sends C_2 to a point off $g(C_1)$? The example on p. 7 of [5] gives an affirmative answer to the second question while the following theorem gives a negative answer to the first.

The following theorem is included for simplicity. It is a special case of the more complicated Theorem 8.4.

THEOREM 8.1. *There is no monotone upper semicontinuous decomposition of S^3 onto itself such that the elements of G are elements of this decomposition.*

Proof. Assume the decomposition can be extended. Then there is a monotone map g of S^3 onto itself such that the elements of G are point inverses under g.

Let J be a simple closed curve in $S^3-g(C_1)$ that cannot be shrunk to a point in $S^3-g(C_1)$. We show that there is no map g by showing that J can be shrunk to a point in $S^3-g(C_1)$.

It follows from the monotonicity of g and the local Euclidean structure of S^3 that for each open set U containing J there is a homeomorphism h of J into $g^{-1}(U)$ such that the identity on J is homotopic to gh in U. We suppose that h is selected for an open subset U containing J such that $U \cap g(C_1)=0$.

Let H_t $(0 \leqq t \leqq 1)$ be an isotopy of S^3-C_1 into S^3-C_1 such that $H_0=$Identity and $H_1(S^3-C_1)=C_2$. Then a homotopy shrinking J to a point in $S^3-g(C_1)$ spends the first part of its parameter pulling the identity on J to gh and the last part as gH_th $(0 \leqq t \leqq 1)$.

EXAMPLE. One cannot weaken Theorem 8.1 by replacing "onto itself" by "onto a subset of itself". If one considers S^3 as the join of circles C_1 and C_2, one sees that there is a monotone decomposition of S^3 onto a disk that homeomorphically sends C_1 to the boundary of the disk and C_2 to the center. Other point inverses are simple curves that link C_1.

DEFINITIONS. Suppose U is an open subset of a polyhedral 3-manifold and J^1 is polygonal simple closed curve in U. We say that nJ^1 *bounds* (mod the integers) in U if J can be regarded as an oriented 1-cycle so that there is some oriented complex C^2 in U such that $\partial C^2=nJ^1$. By adjusting C^2 to remove certain multiple points we may suppose with no loss of generality that it is locally a 2-manifold off J^1 and that at J^1 it locally resembles n sheets with a common edge.

Notice that in considering complexes, we are using the geometric approach rather than an abstract one and regard a simplex as a geometric object (perhaps oriented), a complex as a geometric object (with perhaps its simplexes oriented), and the boundary of an oriented complex as an oriented complex of one lower dimension.

If M^2 is a connected polyhedral oriented 2-manifold in an oriented polyhedral 3-manifold, M^2 locally has two sides. If J^1 is a polyhedral simple closed curve that intersects M^2 in only a finite number of points, it pierces it at each of these, and pierces it n more times in one direction than in the other, we call n the (unsigned) *piercing number* of J^1 and M^2. Similarly, if J^1 intersects a 2-complex C^2 in only a finite number of points, none of these are on the 1-skeleton of C^2, and it pierces C^2 at each of these points, we may define the piercing number of J^1 and C^2 in a similar fashion.

THEOREM 8.2. *Suppose J is a simple closed curve (perhaps wild) in a polyhedral 3-manifold W^3 such that J can be shrunk to a point in W^3. Then for each point $p \in J$ and each neighborhood U of p there is polyhedral simple closed curve $K^1 \subset U-J$ such that no positive multiple of K^1 bounds in W^3-J.*

Proof. Let J be the union of two arcs px_0q, px_1q and B_1^3, B_2^3 be polyhedral 3-cells in W^3 such that $p \in \text{Int } B_2^3 \subset B_3^3 \subset (\text{Int } B_1^3 \cap U)$, $q \in W^3-B_2^3$, and there is a homotopy pulling J into B_1^3 that is the identity on $J \cap B_2^3$.

The 2-sphere Bd B_2^3 is the union of two polyhedral disks B_0^2, B_1^2 such that $B_0^2 \cap B_1^2 = \text{Bd } B_0^2 = \text{Bd } B_1^2$, $px_0q \cap \text{Bd } B_2^3 \subset \text{Int } B_0^2$, and $px_1q \cap \text{Bd } B_2^3 \subset \text{Int } B_1^2$. Then $K^1 = \text{Bd } B_0^2 = \text{Bd } B_1^2$ is the required polyhedral simple closed curve.

Suppose n is positive integer such that nK^1 bounds in W^3-J. There is a polyhedral simple closed curve J_1^1 close to J so that nK^1 bounds in $W^3-J_1^1$, J_1^1 is the union of arcs py_0q, py_1q such that $py_0q \cap \text{Bd } B_2^3 \subset \text{Int } B_0^2$, $py_1q \cap \text{Bd } B_2^3 \subset \text{Int } B_1^2$, and there is a homotopy pulling J_1^1 into B_1^3 that is the identity on $J_1^1 \cap B_2^3$.

We suppose that the homotopy H_t $(0 \leq t \leq 1)$ pulling J_1^1 into B_1^3 is such that there is a finite sequence $0=t_0<t_1<t_2<\cdots<t_k=1$ such that each H_{t_i} is a homeomorphism, $H_0 = \text{Identity}$, $H_1(J_1^1) \subset \text{Int } B_1^3$, and $H_{t_i} = H_{t_{i+1}}$ except on an arc A_i^1 of J_1^1 such that $H_{t_i}(A_i^1) \cup H_{t_{i+1}}(A_i^1)$ bounds a polyhedral disk D_i^2 in $W^3 - B_2^3$. Changing a 2-complex in $W^3-J_1^1$ bounded by nK^1 by adding oriented handles near the polyhedral disks D_i^2 produces in each $W^3 - H_{t_i}(J_1^1)$ a 2-complex bounded by nK^1. In particular, nK^1 bounds in $W^3 - H_1(J_1^1)$.

As a result of the homotopy discussed in the last paragraph, we suppose with no loss of generality that $J_1^1 \subset \text{Int } B_1^3$. Let C^2 be an oriented 2-complex in $W^3-J_1^1$ such that $\partial C^2 = nK^1$. Suppose B_1^2 is oriented so that $C^2+nB_1^2$ is a 2-cycle. (We regard nB_1^2 as n oriented disks all nearly parallel and with the same boundary.) Since J_1^1 misses C^2 and when ordered py_1qy_0p pierces B_1^2 once more in going into B_2^3 than in going out, the piercing number of J_1^1 and $C^2 \cup nB_1^2$ is n, where in lieu of an orientation of W^3 we use an orientation of B_1^3. Let M_1^2 be an oriented polyhedral 2-manifold obtained from $C^2+nB_1^2$ so that J_1 pierces M_1^2 n more times in one direction than it does in the other. We can talk of the direction of the piercing since $nB_1^2 \subset B_1^3$.

Let M_2^2 be a polyhedral orientable 2-manifold in Int B_1^3 such that Bd $M_2^2 = J_1^1$.

We suppose that M_1^2, M_2^2 are in such general positions that their intersection is a 1-manifold with boundary along which M_1^2 and M_2^2 locally cross.

Let $A_1^1, A_2^1, \ldots, A_k^1$ be the components of $M_1^2 \cap M_2^2$ that are arcs. Remove a tubular neighborhood of A_1^1 from M_2^2 so as to change M_2^2 to an oriented polyhedral 2-manifold with boundary M_3^2. There is no assurance that M_3^2 is connected but we suppose it retains the orientation of M_2^2. Certainly, Bd M_3^2 is not connected. The removal reduces by two the number of points in $M_1^2 \cap$ Bd M_2^2. However, since the piercing was from different sides, this leaves the piercing number of Bd M_3^2 and M_1^2 equal to n. By removing tubular neighborhoods about the others A_i^1's, we arrive at the contradiction that there is a 2-manifold with boundary M_{k+2}^2 such that Bd M_{k+2}^2 misses M_1^2 but its piercing number with M_1^2 is n. Hence, our assumption that nK^1 bounds in W^3-J is false.

Note that in proving Theorem 8.2 we made two uses of the fact that J can be shrunk to a point in W^3—first, that J can be pulled into the orientable part B_1^3 of W^3 and second, that J bounds. If W^3 is already orientable we do not need to pull J into B_1^3 and hence obtain the following variation of Theorem 8.2.

THEOREM 8.3. *If J is a simple closed curve in a polyhedral, orientable 3-manifold W^3 and J lies in an open subset U of W^3, there is a polyhedral simple closed curve K^1 in $U-J$ such that no positive multiple of K^1 bounds in W^3-J.*

Proof. As in the proof of Theorem 8.2 we express J as the union of two arcs px_0q, px_1q and let B^3 be polyhedral 3-cell such that $p \in \operatorname{Int} B^3 \subset B^3 \subset U$ and $q \in W^3-B^3$. We express Bd B^3 as the union of two polyhedral disks B_0^2, B_1^2 such that $B_0^2 \cap B_1^2 = \text{Bd } B_0^2 = \text{Bd } B_1^2$, $px_0q \cap \text{Bd } B^3 \subset \operatorname{Int} B_0^2$, and $px_1q \cap \text{Bd } B_2^3 \subset \operatorname{Int} B_1^2$. Then Bd $B_0^2 =$ Bd $B_1^2 = K_1^1$ is a candidate for K^1.

If n is a positive integer such that nK_1^1 bounds in W^3-J, we let C^2 be a 2-complex in W^3-J such that $\partial C^2 = nK_1^1$. Suppose B_1^2 is oriented so that $C^2+nB_1^2$ is a 2-cycle.

Let K_2^1 be a polygonal simple closed curve in $U-J$ so oriented and so close to J that K_2^1 misses C^2, K_2^1 intersects B_1^2 in only a finite number of places, pierces it at each of these, and it pierces B_1^2 once more in going into B^3 than in going out. Let M_1^2 be an oriented polyhedral 2-manifold obtained from $C^2+nB_1^2$ so that K_2^1 pierces M_1^2 n more times in one direction than it does in the other. We shall show that no multiple of K_2^1 bounds in W^3-J—in fact, not even in W^3.

Assume K_2^1 is given an orientation so that mK_2^1 bounds in W^3 and C_2^2 is an oriented complex so that $\partial C_2^2 = mK_2^1$, C_2^2 is locally a 2-manifold off K_2^1, and on K_2^1 it locally resembles m sheets with a common edge. Trim C_2^2 back a bit from K_2^2 to obtain an oriented polyhedral 2-manifold M_2^2 (without singularities) so that the piercing number of Bd M_2^2 and M_1^2 is mn. We do not claim that Bd M_2^2 is connected or even that M_2^2 is connected but we do suppose that M_1^2, M_2^2 are in such general positions that their intersection is a 1-manifold with boundary along which M_1^2 and M_2^2 locally cross.

Let $A_1^1, A_2^1, \ldots, A_j^1$ be the components of $M_1^2 \cap M_2^2$ that are arcs. Remove a tubular neighborhood of A_1^1 from M_2^2 so as to change M_2^2 to an oriented polyhedral 2-manifold with boundary M_3^2. The removal reduces by two the number of points in $M_1^2 \cap \text{Bd } M_2^2$. However, since the piercing was from different sides, this leaves the piercing number of Bd M_3^2 and M_1^2 equal to mn. By removing tubular neighborhoods about the other A_i^1's, we arrive at the contradiction that there is a 2-manifold with boundary M_{j+2}^2 such that Bd M_{j+2}^2 misses M_1^2 but its piercing number with M_1^2 is mn. Hence our assumptions that nK_1^1 bounds in W^3-J and mK_2^1 in W^3 cannot both be true.

EXAMPLE. Theorem 8.3 is false if one omits the orientable requirement. Let P^2 be the projective plane, J_0 be a simple closed curve in P^2 such that P^2-J_0 is an open disk, $W^3=P^2\times(-1, 1)$, and $J=J_0\times 0$. If K^1 is any polyhedral simple closed curve in W^3-J, $2K^1$ bounds in W^3-J.

DEFINITIONS. If B^3 is a polyhedral 3-cell and J_1, J_2 are disjoint simple closed curves in Int B^3, we say that the (unsigned) *linking number* of J_1 and J_2 is n if J_1 is homotopic in Int B^3-J_2 to a polyhedral simple closed curve K_1^1, K_1^1 can be oriented so that it bounds an orientable polyhedral 2-manifold M^2 in Int B^3, and J_2 is homotopic in Int $B^3-K_1^1$ to a polyhedral simple closed curve K_2^1 such that the piercing number of K_2^1 and M^2 is n. It is known that the linking number of J_1 and J_2 is independent of the K_1^1, M^2, K_2^1 selected and is equal to the linking number of J_2 and J_1. We say that J_1 and J_2 *link* if this linking number is not 0. If the linking number is 0, it is possible to adjust M^2 so as to get an oriented polyhedral 2-manifold with boundary M_1^2 so that Bd $M_1^2=K_1^1$ and $M_1^2 \cap J_2=0$.

THEOREM 8.4. *Suppose B^3 is a polyhedral 3-cell in a polyhedral 3-manifold with or without boundary N^3 and C_1, C_2 are linking simple curves in* Int B^3. *Then if G is the decomposition of $C_1 \cup C_2$ whose only nondegenerate element is C_2, there is no monotone upper semicontinuous decomposition of N^3 onto a 3-manifold W^3 such that the elements of G are elements of the decomposition.*

Proof. We show that there is no such decomposition by showing that there is no compact monotone map g of N^3 onto W^3 that has elements of G as point inverses. For convenience we suppose W^3 is polyhedral.

Assume there is such a g. Let U be an open subset of W^3 containing $g(C_1)$ such that $g^{-1}(U)\subset \text{Int } B^3$ and let J^1 be a polyhedral simple closed curve in $U-g(C_1)$ such that no multiple of J^1 bounds in $W-g(C_1)$. That there is such a J^1 follows from Theorem 8.2.

Let h be a homeomorphism of J^1 into $g^{-1}(U)-C_1$ such that $h(J^1)$ is a polyhedron, and the identity on J^1 is homotopic to gh in $U-g(C_1)$.

If $h(J^1)$ does not link C_1, there is an orientation for $h(J^1)$ and an oriented 2-complex C^2 in Int B^3-C_1 such that $\partial C^2=h(J^1)$. Then $g(C^2)$ and a singular annulus joining J^1 and $gh(J^1)$ provides the image of an oriented 2-complex in $M^3-g(C_1)$ whose boundary is J^1. An approximation to this image gives an oriented

2-complex in $W^3-g(C_1)$ whose boundary is J^1. The case where $h(J^1)$ does not link C_1 leads to the contradiction that J^1 bounds in $W^3-g(C_1)$.

There remains the case where $h(J^1)$ links C_1 in Int B^3. Suppose the linking number is n and that the linking number of C_2 with C_1 is m. Let K_2^1 be a polyhedral simple closed curve homotopic in Int B^3-C_1 to C_2. Then $mh(J^1)$ and nK_2^1 each link C_1 mn times. Hence there are orientations for $h(J^1)$ and K_2^1 and an oriented 2-complex C_2^2 in Int B^3-C_1 such that $\partial C_2^2=mh(J^1)+nK_2^1$. An approximation to m singular annuli each joining J^1 and $gh(J^1)$, $g(C_2^2)$, and the image under g of n singular annuli each joining K_2^1 and C_2 provides us with an oriented 2-complex in $W^3-g(C_1)$ whose boundary is mJ^1. The assumption that there was a decomposition of N^3 into W^3 having elements of G as elements led to the contradiction that no multiple of J^1 bounds in $M^3-g(C_1)$ but mJ^1 bounds there.

Question. Would Theorem 8.4 be true if instead of saying that C_1 and C_2 link in Int B^3 we required that they be polygonal simple closed curves in Int B^3 that 1-link there? Recall that two curves 1-link if they do not bound mutually exclusive orientable 2-manifolds.

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